"If it quacks like a sphere"—The Million Dollar Problem

by

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ABSTRACT

Grigori Perelman, a reclusive Russian mathematician, may have proved the Poincaré Conjecture, a statement first poised by Jules Henri Poincaré in 1902. The problem is the most eminent challenge in the mathematical field of topology. Perelman posted his proof on the online informal preprint server at arXiv.org. His proof leaves a number of details unexplained. Although he initially participated in the verification of his proof, answering questions to help people understand his work, in the last year Perelman has effectively disappeared from the mathematical community. His absence has caused some controversy in the world of mathematics, where individual mathematicians are usually expected to support their own results. In the wake of his disappearance, other mathematicians are coming together to pore over his work and try to flesh out the details.

His apparent desertion raises questions both about the personal risk of mathematicians working at the highest level and the responsibility of the mathematical community in the verification process. These questions are further complicated by the fact that the Poincaré Conjecture is one of seven problems that was selected by the Clay Mathematics Institute as a Millennium Prize Problem. If a mathematician solves one of the problems, he or she will receive $1 million from the institute. If Perelman's work turns out to point the way to the prize, then the Clay Institute will have to decide how to distribute both credit and the hefty monetary prize.

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When the directors of the Clay Mathematics Institute (CMI) announced in 2000 that they wanted to give away seven million dollars, the rules seemed clear. They identified seven unsolved problems in mathematics and offered a reward for their solutions. Seven problems, a million dollars each. Solve a problem, get rich.

There barely seemed room for controversy. The CMI intended to award the prizes for accomplishment in mathematics, a discipline associated more with precision and clarity than with ambiguity and compelling human drama. No one expected that the first serious solution that would appear would raise deep questions about both the ambitions—and duties—of the individual mathematician and the responsibility of mathematics community as a whole. But that's exactly what has happened.

For a mathematician, a million dollars represents a superficial reward for years of effort. Significant, but superficial. The establishment of a mathematical breakthrough is no transient victory. A mathematical truth lives forever. The fame associated with these problems transcends the monetary award. The seven individuals or teams who solve one of these problems will achieve eternal fame. Their names and works will be included in the pantheon of great mathematicians.

So far, this exclusive list has remained unpopulated. The Clay roster has been left blank, waiting for the right time, the right work, the right mathematician.

That time might be now. A growing number of mathematicians is convinced that one name should go on the list. According to people in the know, a young and mysterious Russian mathematician named Grigori Perelman might have conquered the Poincaré Conjecture, one of the seven problems.

In its plainest form, the Poincaré Conjecture states that the simplest three-dimensional shape is indistinguishable from a sphere. That may sound trivial, but the conjecture makes a fundamental claim about topology, a branch of mathematics concerned with surfaces. The Poincaré Conjecture has confounded topologists because it concerns the least complicated three-dimensional objects. If you are not sure what you know about the simplest kind of objects, then you cannot be sure what you know about more complicated ones. Thus, the conjecture strikes at the very heart of topology, a field that did not blossom until the 20^{th} century.

"This problem is like the Mount Everest of math conjectures," Australian mathematician Hyram Rubinstein said in a 2003 interview with the Boston *Globe.* "So everyone wants to be the first to climb it." Despite all such efforts, this deceptively innocuous statement has gone unproven for over 100 years.

Grigori Perelman may have changed that.

Though he had achieved a few minor but significant achievements in his early career, Perelman's most notable work had been in the mathematical fields of analysis and geometry. Perelman's work on the Poincaré Conjecture, which resides in the mathematical realm of topology, came from nowhere and caught the mathematical community off guard.

"Nobody suspected what happened until it happened," said Jim Carlson, president of the Clay Institute. "It was a surprise and a delight."

If validated, a proof of the Poincaré Conjecture will confirm over a hundred years of mathematical investigations into three-dimensional geometric objects. Scores of mathematical results depend fundamentally on its truth, and until it is proven, these results remain speculative.

Given these high stakes, the proof of the Poincaré Conjecture has attracted and subsequently eluded generations of mathematicians. John Stallings, a mathematician at Berkeley, was even compelled to write a short paper about his efforts. "I have committed the sin of falsely proving Poincaré's Conjecture," he wrote in a paper titled "How not to prove the Poincaré Conjecture."

Stallings continued, "But that was in another country; and besides, until now no one has known about it. Now, in hope of deterring others from making similar mistakes, I shall describe my mistaken proof. Who knows but that somehow a small change, a new interpretation, and this line of proof may be recited!"

His paper outlines the mistakes that ensnared him as he pursued a proof. While Stallings's list is highly technical, he writes that his chief mistake was to try to simplify, "by some geometric trick," the big picture. But every apparent simplification hid a deeper and unsolvable complexity.

"This phenomenon has characterized every attempt that I have made or heard of to prove Poincaré's Conjecture," he wrote. He concluded, "I was unable to find flaws in my proof for quite a while, even though the error is very obvious. It was a psychological problem, a blindness, an excitement, an inhibition of reasoning by an underlying fear of being wrong. Techniques leading to the abandonment of such inhibitions should be cultivated by every honest mathematician."

"Almost every false proof… has stumbled on one of the mistakes that he [Stallings] lists," said Tomasz Mrowka, a mathematician at the Massachusetts Institute of Technology who has been following Perelman's work for years.

"Many people have tried to solve it, mostly using ideas within topology, and they've all failed," Carlson said. "There have been people who announced a proof, and then it's fallen apart within a matter of days. When a proof is faulty, it's usually discovered to be faulty very quickly."

The most recent failed attempt was by the well-respected British mathematician Martin J. Dunwoody in 2002. His proof was only five pages long, and was found, within a matter of days, to contain a crucial mistake.

Perelman's proof, so far, has withstood scrutiny for over three years.

In late 2002 and early 2003, Perelman submitted a series of three papers to the online preprint server at arXiv.org. The arXiv, which is hosted by the Cornell University Library, is an anarchic graffiti wall for the intelligentsia of the hard sciences. Mathematicians, physicists, computer scientists and quantitative biologists check it daily and post their own work there. Some of the posted papers are peer-reviewed and have appeared in journals; others, like Perelman's, are rough drafts in search of peer editors and evaluators. In a combination of mathematical language and prose, Perelman's papers introduced an unexpected approach to the conjecture. He applied methods first developed by the American mathematician Richard Hamilton in a way that no one had ever seen before. Mathematicians say that his work is both intensely dense and remarkably creative.

"What he writes is sort of about as little as you can say and actually say everything you were supposed to say," said Mrowka. "Besides the new ideas, he really mastered a lot of area around it. He knows how to say the arguments very quickly and efficiently.... There's a lot of new, very cool stuff in those papers. He saw a lot of things that people hadn't seen."

In 2003, news of Perelman's work spread through the mathematical community and appeared in a number of periodicals. In addition to the *Globe* story, the *New York Times* and the BBC ran articles that suggested the conjecture had been proven. The stories ran before anyone had really had a chance to look at the proof; the verification process may take years.

If verified, Perelman's work will not only prove the Poincaré Conjecture; it will prove a much larger statement about three-dimensional objects called the Thurston Geometrization Conjecture. (It is also referred to as the geometrization conjecture.)

In a paper that summarizes the ideas contained in Perelman's papers, mathematicians Bruce Kleiner and John Lott wrote that Perelman's work contains a "wealth of results about the Ricci Flow," the mechanism used by Perelman. His insights and methods were so innovative that Perelman seems to have achieved the apex of mathematical accomplishment: he has placed his imprint on an entire field.

In order to understand what Perelman did, one first must understand the problem. For a math problem, the Poincaré Conjecture is relatively straightforward.

A conjecture, in math-speak, is a statement that is generally believed to be true but has not yet been proven. In a 1904 paper, the French mathematician Jules Henri Poincaré said that he believed that any shape that shares a certain characteristic with an ordinary sphere is also a sphere. This commonality is called "simple connectivity," and will appear later. (A sphere is most often defined as a collection of points equidistant from a common point, the center.) For right now, however, think of it this way: Poincaré said that, no matter what it looks like, a sphere is a sphere is a sphere. If a shape acts like a sphere, behaves like a sphere and tastes like a sphere, then it might as well be a sphere.

"In other words, if it quacks like a 3-sphere…" Carlson said.

A sphere is a sphere is a sphere. You can punch, kick and throw it to your heart's delight. You can inflate it and deflate it. You can mold the sphere into another shape. You can make depressions in it with your fingertips all the way around to create patterns that Martha Stewart might admire (or abhor). In short, you can do your best to make the shape look like anything but a sphere. And these deformed, twisted and complicated shapes that you end up with, in the wild world of topology, are still spheres.

What you cannot do, however, is poke a hole in it. You cannot, for example, turn your sphere into a donut. Or a donut with two holes. You cannot turn it into a coffee cup with a handle, frames for your eyeglasses or a key ring. You can stick your finger into it, but you can't actually puncture the surface or reach inside. Don't even think about sticking it with needles, cutting it with scissors or climbing inside of it. You absolutely, positively are not allowed to break the surface in any way. To break the surface of the sphere is to venture into a different group of topological objects.

A sphere is a sphere is a sphere. Say you're walking down a street, and you encounter a strange and complicated shape whose surface presents peaks and valleys, mountains and molehills (but no holes, remember). And you want to know more about it. As a mathematician, you want to study the way that functions behave on it. Poincaré's conjecture said that no matter what it looks like, it's a sphere, and if you want to analyze it mathematically, you just have to play with the math you would have done on a regular sphere. Poincaré's conjecture gives mathematicians a short and easy way to identify a deformed blob as a sphere in disguise.

There is one more complication. When most people think of a sphere, they generally consider the space that a sphere occupies. A sphere in three dimensions. A ping pong ball, for example, or the earth. When topologists talk about a sphere, they are talking exclusively about its surface. As we think about a sphere as being a ball, the exterior together with everything in its interior, a topologist would think of a sphere as being just the outside of the ball.

A 1-sphere, for example, is the outside of a circle. A 2-sphere is the curved surface of what we consider a sphere. It is two dimensional because, if you stand on the surface and look around, it appears that you are in a two dimensional space. The surface of the earth serves as a rough analogy—the world essentially appears flat, when we stand on the ground and survey the horizon.

Since the surface of the Earth might be considered to be a 2-sphere, we will not change its topological 'sphereness' if we deform it. Imagine deflating and pinching and reshaping the surface of the earth until it was unrecognizable as a sphere. If the 2-sphere on which we live were contracted, deformed and wadded, it might be possible to walk out your front door in Brooklyn and end up in Hawaii. Or Cleveland. It might be possible to roll a bowling ball in Kansas City and score a strike in Hong Kong. The Poincaré Conjecture essentially said that it doesn't matter if you wad it or deform it, dilate or contract it; a sphere is a sphere is a sphere.

The above speculations explore the characteristics of a 2-sphere; in its original form, however, the Poincaré Conjecture concerned three-dimensional spheres (i.e., 3-spheres). These shapes are difficult—perhaps impossible—to visualize, but mathematicians are not concerned with its appearance. They are concerned with its characteristics.

You may not know what it looks like; you may not be able to draw it for your children. You may not even be able to dream about it, live in it, or vandalize it. Nonetheless, do what the mathematicians do. Imagine that if it exists in two dimensions, then we can extend our imagination and knowledge to believe it exists in three dimensions. Even without being able to picture it, draw it, or know that it exists, we can do math on it. We can calculate distances between points (this exercise is at the heart of complex data analysis.) Topologists dedicate their entire careers to these sorts of thoughts. They thrust their brains into new dimensions, working with abstractions.

It may be impossible to visualize a 3-sphere, but it is possible to explore it. Any system that can be characterized by three numbers automatically determines a three-dimensional shape. Consider the weather as an example. If you take multiple measurements of the temperature, the humidity and the wind speed, you will end up with a collection of data points that are determined by three numbers. These numbers might be thought of as coordinates, and then you've determined a "weather shape" that is three-dimensional. In baseball, if you tally the numbers of runs, pitches and fouls for each inning of a game that doesn't go into overtime, you have established 9 data points in a three-dimensional space. With those nine points, you could make statements about the "shape" you have created.

You could ask even more complicated questions about the implications of a 3-sphere. What if we lived on the surface of a 3-sphere? If our universe, for example, was a not-too-impossibly-large 3-sphere (a scenario seriously considered by physicists), we'd be able to shoot rockets deep into space, and eventually, the rockets would come back to us from the opposite direction. If you went far enough from where you started, you'd end up where you began.

Now imagine that this 3-sphere universe, like our poor balloon, is distorted, wadded, dilated, and deformed (but not punctured). If we lived in this deformed 3-sphere, then you could feasibly walk across the Golden Gate Bridge and end up on Mars. Or on the event horizon of the massive black hole holding the Milky Way together. You might find yourself on another planet, somewhere in space, where alien beings have such a solid understanding of the topology of the universe that they are not at all surprised to see you.

The Poincaré Conjecture offered a way to identify these blobs as spheres in disguise: it said that if a blob really is a sphere in disguise, then it must be simply connected. In topology, "simply connected" means that you can draw, anywhere on the surface of the thing, a circle, and you can contract that circle to a point. A ping pong ball is simply connected: you can imagine drawing a circle, and shrinking it to down until it was just a dot.

Or, you could think about it this way. You tie a lasso around your blob and tighten it until the string lies on the surface. If, for *every* different way you can tie the lasso, you can slip it off, then the blob is a sphere. If it is possible to tie the lasso in such a way that it proves impossible to remove the lasso without breaking either the rope or the blob, it is not a sphere.

A donut, for example, is not simply connected. It is possible to draw rings on a donut that cannot be contracted. Imagine that an ideal Krispy Kreme delicacy—immediately out of the sugar shower, hot, of course—is a perfect torus. Now imagine a single chocolate ring that begins on the outside of the donut, crests the top of one side, goes through the center hole, and winds up where it began. As delicious as this ring may be, it is impossible to contract. If your lasso passes through the center of a donut, you cannot remove it without either altering the shape of the donut or cutting the rope.

Though this is not its mathematically precise term, 'breaking the donut' is absolutely, positively not allowed in the world of topology. You may say that, above all things, topologists respect the donut.

In mathematical terms, the Poincaré Conjecture is that every simply connected 3-dimensional shape is homeomorphic to a 3-sphere. "Homeomorphic," in this case, means "topologically equivalent"—i.e., indistinguishable to topologists. Spheres in disguise are still spheres. In other words, if you can, without breaking the shape, mold it into another shape, then the two are homeomorphic. A basketball, for example, is homeomorphic to a football. An apple is homeomorphic to an orange.

If an object has a hole that passes all the way through it, then it is not homeomorphic to a sphere. In the class of two dimensional objects that contain one hole, a bicycle tire is homemorphic to a key ring. A coffee cup is homeomorphic to a donut. In fact, a favorite joke among mathematicians is that topologists cannot tell the difference between a coffee cup and a donut.

And this is why Perelman's story is so compelling. Perelman is not, by training, a topologist. He can distinguish a coffee cup from a donut. But he stepped into the twisted world of topology and found gold on his shoe.

To illustrate the surprising aspects of Perelman's work, Jim Carlson draws a very complicated squiggle.

"This is really a circle, but it's a very wild circle," he says. If his squiggle were an island, it would be almost impossible to dock a boat on its shores. "What you do is draw a very complicated curve like this with lots of little inlets and outlets, and then you go back and you close it up and then you ask yourself, is this point right here"—he makes a dark mark somewhere within the ink labyrinth—"is this point inside or outside?"

Carlson's circle looks like it's taken quite a beating. Mathematicians would say that Carlson's 'wild circle' is smooth because you can trace the entire figure without lifting your pencil. But smoothness does not imply simplicity, and the complexity of the design suggests that his question will require a considerable amount of time to figure out.

"You could trace the maze and eventually figure this out, or—"he marks a heavy dot on the paper outside the sketch—"it turns out there's a quick way to do it. You draw a line from that point to a point you know is outside—because it's very far away—and you count whether the number of crossings is even or odd, and if it's even, it's inside, if it's odd, it's outside."

Carlson is using his squiggle to demonstrate a very simplified version of a test that topologists use to determine whether or not an object is simply connected. Not content to work with prosaic, real-world objects like donuts and coffee cups, except as lower-dimensional examples, topologists study simple connectivity as it applies to objects of four, five, six or more dimensions. The elusive, stubborn case of the Poincaré Conjecture, however, has always been rooted in three dimensions.

Carlson returns to his squiggle to explain Perelman's methods.

"The idea is, in some sense, to apply heat to the shape and to allow the heat to… simplify it. Take this very complicated wild circle, and imagine putting a little air hose in here and inflating it," he said, drawing a little box near the labyrinth. "It will dilate, and eventually it will achieve a round shape. Imagine a crinkled up balloon – you want to know what its real shape is, well blow it up with air, and then look at it. It achieves the simplest possible shape after you blow it up enough," he said.

Poincaré's conjecture challenged mathematicians to prove that the simplest 3-dimensional manifold is, in fact, a 3-sphere. Perelman's proof of this statement requires the successful introduction of air, or heat.

Imagine you have procured a box whose colorful label announces that it contains an inflatable raft. When you get home and unpack the box, you find inside a dense pile of rubber folded into the shape of the box. It hardly looks like the raft you were promised. Undaunted, you attach an air pump to the small valve protruding from one side and turn on the pump. As air enters the rubber, the blob unfolds slowly and begins to resemble a raft. As more air enters the raft, it swells and bulges, and before long the pile of rubber has evolved into a raft identical to the picture on the packaging. With fast hands you remove the pump and close the valve.

This raft, by the way, assuming it's not a tube, is homeomorphic to a sphere. Now you and your sphere-disguised-as-a-raft can head to the beach.

Of course, topologists do not need a valve. (They could not use it, anyway, as it would violate the rule that one must respect the sphere and not poke holes in it.) Instead, they take the 'what if' approach. What if one could, without breaking the surface of the object, introduce an air field? What would happen to the surface if you allowed it to assume its simplest shape? This theory—of adding air, or heat, to a complicated shape—was first developed by Richard Hamilton in the 1980's and is called Ricci Flow.

Hamilton came close to solving the Poincaré Conjecture himself. "Hamilton came to the Ricci Flow as, I think, a wild guess as to something that looked like a sensible equation to study," Mrowka said.

Carlson agreed that Hamilton deserves an enormous amount of credit for the solution. "He [Hamilton] was the one who came up with the idea that drives the entire program; that was 15 or 20 yeas ago. Inventing this equation is already a huge step forward, and he proved many things about it. He greatly advanced the subject. He's been the prime mover in the subject," he said.

In his work on the Poincaré Conjecture, however, Hamilton failed to successfully account for singularities that may arise on the object. Singularities might be thought of places where the fabric of the object is 'pinched.' Imagine that the air mattress, for example, turned out to be shaped like a barbell. In that case, the two sides of the barbell would continue to inflate to spheres, while the connecting rod became thinner and thinner; as time goes on, it will not resemble a sphere. This connecting rod would be considered a singularity in the shape.

Perelman's major step forward was to essentially 'snip' the rod. By utilizing these abstract scissors, Perelman's method allowed each side of the barbell would become its own sphere. The two resulting spheres would be topologically indistinguishable. Mathematicians refer to this process as 'surgery' on a 3-dimensional object. Perelman's use of surgery on these complicated surfaces was unprecedented, unexpected, and derived from his expertise in seemingly unrelated mathematical fields.

"The interesting thing about the Poincaré conjecture is that the proof is really coming from outside topology—it's coming from geometry and analysis," Carlson said. If topology is the study of surfaces, then geometry is the study of shapes and their properties, and analysis the study of how mathematical functions behave on these shapes and surfaces. As an outsider to the field where he solved a major problem, Perelman joins a distinguished tradition in mathematics.

In 1993, Andrew Wiles, a British mathematician, presented a proof of an unsolved problem known as Fermat's Last Theorem. His work, like Perelman's, demonstrates that in mathematics, the most innovative and dramatic advances often come from unexpected places. Wiles's proof of Fermat's Last Theorem used functions of elliptic curves, a branch of mathematics that had not been developed when Pierre de Fermat wrote his famous statement sometime around 1630.

"It's a huge surprise when you discover that the question you were asking was not the right way to ask a question, so you have to think about it in a different way," said John Baez, a mathematical physicist at the University of California-Riverside. "Math is calling for you do to certain things; it has its own grain to it and demands to be cut in certain ways."

Fermat's problem arose from a note that he had written in the margin of his copy of *Arithmetica* by Diophantus of Alexandria. After explaining the problem, Fermat wrote in Latin: "I have a truly marvelous proof of this proposition which this margin is too narrow to contain." For more than three centuries, mathematicians sought Fermat's famously 'marvelous' proof, to no avail. And others wondered whether a proof really existed, or whether it was correct, or whether Fermat was, in fact, leaving the world with one irresistible and abstract practical joke.

The allure of a possible proof of Fermat's Last Theorem has leaked into mathematical humor. A common joke goes like this: A mathematician was invited to give a talk at a conference. The conference organizer asked the mathematician, "what will you be talking about?"

The mathematician replied, "I will be proving Fermat's last theorem."

"Fermat's last theorem!" the organizer exclaimed. "This is history in the making!"

On the day of the conference, the mathematician addressed the anxious crowd with his latest result, which had nothing to do with Fermat. Nothing. Not even a mention. After his talk, the organizer said, "I thought you were going to prove Fermat's last theorem!"

The mathematician said, "Oh, that. I always say that, just in case I get in a car wreck on my way here."

That joke has expired, thanks to Andrew Wiles. In a television interview, Wiles recounted his first encounter with Fermat's statement.

“This problem had been unsolved by mathematicians for 300 years,” he said. “It looked so simple, and yet all the great mathematicians in history couldn't solve it. Here was a problem, that I, a ten year old, could understand and I knew from that moment that I would never let it go. I had to solve it.”

Wiles worked on the problem for his entire career, and towards the end of his quest he isolated himself, not telling his friends and colleagues what he was up to. He was diligent and secretive. In 1993, he revealed the results of his labors and stunned the math world.

Shortly thereafter, his proof was found to contain a critical error, and Wiles went back to work. After going back through his work with his former student Richard Taylor, Wiles emerged again, this time with success. His proof still stands.

Wiles's proof, however, is almost certainly not the proof that Fermat had referenced in his mystifying note.

“I don't believe Fermat had a proof,” Wiles said. “I think he fooled himself into thinking he had a proof. But what has made this problem special for amateurs is that there's a tiny possibility that there does exist an elegant 17th-century proof.”

The story of Wiles and Fermat's Last Theorem shows the intensely personal level that a mathematical quest can take for an individual. There are other stories like Wiles's sprinkled through the history of mathematics that illustrate what the quest for solutions can do to people. The historians of science Loren Graham and Jean-Michel Kantor, in a recent paper, remark upon the influence of “mystical religious views” on a group of Russian mathematicians in the early 20th century. During the same period, Graham and

Kantor report, a group of French mathematicians observed that studying transfinite numbers "might cause some mental disturbances."

A bright mathematician, whose talent borders on obsession, sees an unsolved problem and gives chase. To win the struggle he or she must face and conquer uncertainty, remove doubt. It's like climbing a mountain, just because it's there. Solving an unconquered problem is a victory in the abstract frontiers of human understanding.

The summons proves irresistible to certain types of people. It always has.

From Diophantus of Alexandria, who in the third century CE enumerated a list of unsolved (some unsolvable) problems in his *Arithmetica*, to the Clay Institute's contest, mathematicians have always been interested in identifying both the problems that will shape the science, and the solutions to those problems.

"We hear within us the perpetual call: there is the problem. Seek its solution," David Hilbert, a young and famous German mathematician, said in August 1900 to a crowd gathered in Paris for the second meeting of the International Congress of Mathematicians. His dictum seems almost spiritual in tone.

Hilbert continued, "You can find it by pure reason, for in mathematics there is no *ignorabimus*." *Ignorabimus*, translated from Latin, means 'we will not know.'

Hilbert set the precedent for the Clay Institute's competition. At his historical talk, Hilbert articulated a list of problems that he believed would guide the evolution of mathematics in the 20^{th} century.

"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?" Hilbert asked.

At 38, Hilbert was well-established; he had made a name for himself with significant contributions to algebra, geometry and number theory. He is often referred to as one of the greatest mathematicians of the 20th century. In preparing for his speech, he consulted with his friend and fellow mathematicical great Herman Minkowski. Minkowski wrote to Hilbert, "Most alluring would be the attempt to look into the future, in other words, a characterization of the problems to which the mathematicians should turn in the future. With this, you might conceivably have people talking about your speech even decades from now."

Hilbert took Minkowski's advice. His speech was a rallying cry to the mathematicians of his own generation and to generations of mathematicians yet to come. He spoke eloquently of the need for mathematical solutions to be clear; he quoted an 'old French mathematician' who said "A mathematical theory is not to be considered until you have made it so clear that you can explain it to the first man whom you meet on the street." He spoke to the wandering nature of mathematical solutions. One solution could lead to new mathematical landscapes and ideas.

"...A mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts," he said. "It should be to us a guide post on the mazy paths to hidden truths, and ultimately a reminder of our pleasure in the successful solution."

Although he only had time to speak about 10 problems, Hilbert published a list of 23. Hilbert's problems became so ingrained in the mathematical canon that Herman Weyl, a student of Hilbert's whose work brought together analysis, geometry and topology, wrote that a mathematician who solved one of them "passed on to the honors class of the mathematical community." In a 1951 article, Weyl wrote, "How much better he predicted the future of mathematics than any politician foresaw the gifts of war and terror that the new century was about to lavish upon mankind!"

In the introduction to his book *The Honors Class*, Benjamin Yandell wrote, "Hilbert's list is a thing of beauty, and aided by their romantic and historical appeal, these well-chosen problems have been an organizing force in mathematics."

The list has inspired many versions of mathematics "best-of" inventories. At the ICM meeting in 1998, Fields medalist Steven Smale articulated a list of 18 problems that he believed would advance 21^{st} century mathematics. In 2000, as part of a promotion of a new book, the London publishing firm Faber and Faber offered $1 million to anyone who could prove the Goldbach Conjecture, which states that any even number greater than four can be written as the sum of two prime numbers. Their prize went unclaimed, and the publicity stunt closed in 2002. The Clay Institute honored Hilbert's legacy by announcing the Millennium Prize problems in Paris, 100 years after Hilbert enumerated his list.

The Poincaré Conjecture, which did was not formulated until 1904, was not on Hilbert's original list, but its solution will undoubtedly govern the future evolution of the field of topology. And the name associated most directly with the proof will be Perelman's.

The first two of the three papers published at arXiv.org contain the bulk of Perelman's work. In 2003, following the initial excitement his work caused, the reclusive Russian came to the United States and gave a series of talks at Stony Brook, the Massachusetts Institute of Technology, and Princeton University.

The more Perelman talked, the more people took him seriously. And the more apparent it seemed that a proof for the Poincaré Conjecture existed.

"When he gave these talks, when people raised all kinds of questions, he was able to give very cogent answers that satisfied the people who asked the questions – that's a very important sign," Carlson said. "It's not a proof that things are correct, but it is a sign that things are going in the right direction."

MIT's Mrowka attended Perelman's lectures. He said that Perelman's talks went a long way in convincing mathematicians of the merits of his work. "While he's there, you can ask him, can you explain this point, this sentence, he'll give you extremely clear explanation, which if you wrote down would take several pages," Mrowka said.

Carlson said that this sort of mathematical dialogue is absolutely crucial to the acceptance of a proof by the mathematical community. "If a person cannot respond satisfactorily to questions, that's a big danger sign, and if they don't put themselves in a situation where they can be questioned, that's also a big danger sign," Carlson said.

Perelman rose to the occasion. At seminar after seminar, school after school, he successfully fielded question after question. At the same time, he refused to be interviewed by journalists, and he snapped at a photographer who tried to take his picture.

If Perelman's proof is verified and the Poincaré Conjecture conquered, then he stands to not only earn a place in the hall of fame; he also stands to collect that hefty monetary prize.

But, something strange has happened.

"In the beginning, [Perelman] was quite responsive to people's email questions but apparently, in the last year or so, he has been less communicative," Carlson said.

Perelman, on the brink of becoming the biggest celebrity in the math world since Wiles proved Fermat's last theorem, returned to the Steklov Institute in St. Petersburg, Russia. He stopped collaborating with people who were anxious to understand his work. He was invited to seminars and colloquia designed to verify his proof, but he did not participate.

Perelman, in the eyes of the math world, has vanished. And the work he left behind is incomplete. Like a treasure map, his papers from the arXiv may point to a correct proof of the Poincaré Conjecture, but they leave many details unexplained. His vanishing act upstages even that of Fermat: not only does he claim to have a proof, but he's left an outline.

His disappearance from the math world has also caused an unusual dilemma for the CMI, a dilemma that speaks to the nature of authority in the mathematical academy and strikes at the definition of the word 'proof.' They have before them a potential proof of one of their seven problems, but without his participation it has become hard to know how, when—or whether—anyone will be able to conclude that the proof is in fact true.

Their dilemma is this: since Perelman published his work informally, on the Internet, and not in a journal, his work, as it now stands, is not eligible for the prize. In their description of the Millennium Prizes, the rules are very clearly laid out and include words like "peer-reviewed" and "refereed." Normally, these are not unusual expectations. Normally, this aspect of math—the verification side, the more institutional side—is fairly prosaic.

The process of mathematics might be thought of as having two components. One embraces the wildly creative impulses that allow people like Wiles and Perelman to reach across disciplines and suggest solutions to elusive problems. The other encompasses the verification process, the methods by which mathematicians talk to each other and verify the truth of a given proposition. If there is any semblance of scandal to be found, it usually resides in the former, more tempestuous arena.

But the strange case of Grigori Perelman is a different matter altogether.

When a mathematician disappears before the world is ready for him to do so, death is usually the culprit. Fermat's last theorem gained notoriety because he died before be could either supply his proof or admit he had made a mistake. Evariste Galois, whose brilliant contributions to the theory of equations were not recognized until long after his death, died in a duel—allegedly over a woman—at the age of 20.

Perelman hasn't died, but his disappearance has left people asking questions.

"It's very strange," MIT's Mrowka said. "It's unprecedented that somebody has taken so little interest in the verification of their work. Usually, if somebody makes some big claim, they are a big part of checking it, and that helps a lot."

Perelman has checked out, and his silence raises questions about both the personal and social aspects of mathematics. If a mathematician writes a proof, but he does not fill in all the missing steps, did he really *prove* anything? At what point does a proof become truth? Who, after all, finally decides when a proof is valid?

"Where does the responsibility lie for convincing people that this is a correct proof?" asked Barbara Keyfitz, director of the Fields Institute for Mathematical Research in Toronto. "What is the relative responsibility of the person who claims to have proved it, and of the people that are listening to them, to figure out these puzzles that remain?"

Perelman's proof has left plenty of puzzles, not least of which is its correctness. There are some simple proofs—geometrical proofs, for example, the ones that every high school student either loves or loathes—that are self-evident in their truth. They appeal to reason; there is something intrinsically "right" about correct proofs. Mathematicians often use the word "elegant" to describe them.

And then there are the more complex proofs.

It is impossible to look at Perelman's work, in all of its terseness, and see its legitimacy. In verifying Perelman's proof, mathematicians hope to remove the last traces of doubt from the truth of Poincaré Conjecture. This is a worthwhile venture—lots of insightful, creative mathematics produced over the last 100 years depends on the conjecture. The work resulting from Perelman's ideas, when all the questions have been answered and all the nuances explained, will fill hundreds of pages. But will they still be attributed to him? Should they be?

The answers will not be known for years. Since Perelman has gone mute, others are rushing in to make sure he's right. And when—and if—the proof is published, it will be collaborative. At least two independent groups of mathematicians are currently working on supplying the missing details of Perelman's opus, and Carlson said their work should produce peer-reviewed, or refereed, books and articles. The CMI has organized seminars and colloquia with the intention of bringing together eminent mathematicians to scrutinize Perelman's work. Two years ago, they organized a meeting at Princeton University to look over part of Perelman's proof. The participants included Yu Ding, Bruce Kleiner, John Lott, Peng Lu, John Morgan, Natasa Sesum, Gang Tian, and Guofang Wei.

"The idea there was to invite people who already had spent considerable time working through Perelman's first paper and to go through, pretty much line by line, section by section, Perelman's second paper," Carlson said. "It was intellectually very intense – people were working very hard. By the end of that workshop, I think people felt pretty confident about the proof." Carlson added, "Perelman was invited, but he did not attend."

"It may not be that Perelman's original papers will be the subject of the direct refereeing process," Carlson said. It will be "a little more complete account that involves the work of these other people will be published at some point."

As far as Carlson is concerned, Perelman's participation is less important than establishing the Poincaré Conjecture as truth.

"There's a standard in mathematics. We don't do experiments. We do proofs. There has to be a complete proof written down, published and refereed" before the CMI can award the million dollars.

So the proof, when it is finally published—"somewhat fairly soon," Carlson said—will be based on Perelman's work, but the authors of the paper will be the mathematicians who are participating in the evaluation.

"Once there is a refereed publication, then the clock starts for the Millennium Prize," Carlson said.

The 'clock' will run for two years. After two years, the Scientific Advisory Board of the CMI will appoint another group of experts to assess whether or not the proof has held up. "They [will] look at the result of the refereeing process, they would independently look at these publications that have been refereed, and they would make some sort of recommendation, and then the scientific advisory board would either recommend that the prize be given or not, based on the evidence at that point."

At that point, assuming everything checks out in Perelman's work, the board of the CMI will have to decide how to divide the prize. But besides the practical considerations, there is a fundamental question that underlies the CMI's process of 'going around' Perelman: who determines when a proof is completed? There is no governing board, no committee that gives a thumbs-up to a mathematician truth. Rather, as the CMI's lengthy and rigorous process seems to suggest, an invisible consensus seems to grow in the community.

Respectability and scrutiny generally come from the many peer-reviewed mathematics journals that circulate through universities and institutes around the world. Even after a result—this is how mathematicians refer to a significant conclusion, as a 'result'—appears in a journal, it must withstand the test of time. As more people study the proof, they will better understand it; as they understand it, they'll use Perelman's math in new and unexpected applications.

"The thing that really is more convincing … is, can people go beyond what the claims are and use the tools to do something else?" asks MIT's Mrowka. "People will clean up the foundations better and see how to get to the goal more quickly, and as these things get cleaned up you get more confident…. There's … a clear level of rigor that people accept."

So the Perelman case presents a novel problem—since the author himself is not making any effort to present his work to a peer-reviewed journal, the work that is eventually refereed will, in essence, be a group effort. To some mathematicians, that process raises significant questions about mathematical creativity and responsibility.

At their annual meeting in February, the American Academy of the Advancement of Sciences (AAAS) hosted a panel discussion titled "Paradise Lost? The Changing Nature of Mathematical Proof." The panelists discussed the ways in which mathematical developments have effectively shaped the very process by which mathematicians determine truth.

One of the panelists, Washington University mathematician Steven Krantz, criticized the process by which the mathematical community is accepting and celebrating Perelman's work. Perelman's solution has not been verified or published yet, he said, but the mathematical world assumes, without justification, that it will check out. The proof is not yet complete; it is not yet a proof. It contains leaps in logic that may require years to bridge. He told the audience that he had offered to publish Perelman's detailed work in *The Journal of Geometric Analysis*, but his offer was met with silence.

"It is now up to the mathematical community to make of Perelman's work what it can," Krantz told the crowd. "According to my value system, if you think you have proved a theorem then you write down a proof so that others can check it and understand it. You circulate your work among authorities so that it can be refereed and verified. This new proof of the Poincaré conjecture has sidestepped all these fundamental principles. It has left an entire subject in limbo."

Krantz added, "I think that one of the most important aspects of our discipline is verification and archiving. The new program to prove the Poincaré conjecture thus far is sorely lacking in this respect. It is counterproductive, it is irresponsible, and in the end it is discouraging for us all."

As Krantz has pointed out, traditionally, it is up to a mathematician to fill in his own gaps and let his peers evaluate his work. But since Perelman has disappeared, others are doing his work for him. Krantz's challenge to the process points to pivotal questions regarding the nature of mathematical acceptance. How much proof does a mathematician have to supply? What is the responsibility of a mathematician?

"There is a level at which you start saying, 'I'm not giving you all the details, but the details wouldn't be hard to work out – there's something there, it's routine, I did it for that case, it's going to be exactly the same for this case,'" said Keyfitz. She uses the word 'mystical' to characterize the ways that mathematicians make such associations. "This case is going to be the same as that case, so I don't really need to show you the details."

"Most proofs that mathematicians produce are not formal," said Keith Devlin, a Stanford mathematician and writer who has gained a wide public following as the 'math guy' on National Public Radio. Devlin was also a member of the AAAS panel. "They omit steps that they regard as 'obvious' to themselves and to the intended audience. You tailor the argument to match your audience. In principle, a reader can always fill in all the missing steps. But this is completely impossible to do in practice."

This is exactly Krantz's problem with Perelman—he has not filled in all the missing steps, a crucial element in the verification process.

But this job—filling in all the steps—can be a Herculean task. For example, there exists an epic mathematical quest to classify the "finite simple groups." Like the

Poincaré Conjecture, this problem appears to be fairly simple. It is one of the fundamental ideas behind group theory, a branch of mathematics concerned with the different ways to organize sets of objects. The theorem to be proved says that every finite set of objects you can imagine—sets of numbers, of letters, of people, it doesn't matter—belongs to one of just five distinct categories.

The first step in proving this claim arrived in an 1899 paper by William Burnside, and mathematicians continued to work on the theorem for almost 100 years. The last classification paper appeared just a few years ago. The pursuit of this puzzle spans the entire 20th century, and its solution contains the contributions of hundreds of mathematicians.

The word 'simple' in the problem's name might suggest that a proof would be fairly straightforward, fairly compact. Not so. According to Michael Aschbacher of CalTech, another panelist at the AAAS meeting, current estimates place the length of the proof at more than 10,000 pages. Appropriately, the problem of classifying finite simple groups is also known as the "Enormous Theorem."

But the work seems to be done; mathematicians believe the theorem has been proved. Aschbacher said that "all the known gaps have been filled."

The pursuit of mathematics is no stranger to collaboration, especially in the last few decades. As mathematicians are able to communicate with each other more quickly, they are able to work with international collaborators. The case of the finite simple groups is just one example of high-level group work.

"Nonetheless," Carlson said, "I would say that some of the most important contributions—if not many, or perhaps even most—are made by individuals.

[Perelman's work is] built on the work of other individuals like Hamilton; it's likely to end up taking a group of individuals to have verified it and worked out all the details; but that's the case with any mathematical result."

He continued, "There always has to be at least a small community of people who seriously read the paper and think about the result, try to poke holes in it, and come to the conclusion that it's solid or not."

In order to verify Perelman's work, a group of mathematicians must be willing to examine it thoroughly. But with so many holes, so many questions left unanswered, why would mathematicians take it upon themselves to finish the work? Perelman's case intrigues the mathematics community because it suggests that they may be closer to understanding a fundamental aspect of three-dimensional objects. So, despite the Russian's strange behavior, the tantalizing prospect of proving the Poincaré Conjecture is irresistible. His work matters.

"Suppose somebody does submit something, even in an unconventional way, people will jump on it," Carlson said.

The interest of the mathematical community in Perelman's work illustrates a critical feature of the social forces at work in the process of the mathematical community. Because his work *appeared* to be correct, people took it seriously. That is not the case, however, with every proof that appears online.

Louis de Branges de Bourcia, a mathematician at Purdue University, claims to have proven the Riemann Hypothesis, perhaps the best known of the Millennium Prize Problems. But, unlike Perelman, de Branges has been largely ignored.

The Riemann Hypothesis is considered by some to be the Holy Grail of contemporary mathematics. It has been around, tantalizing mathematicians, for quite a while. On David Hilbert's list of problems from 1900, the Riemann Hypothesis was #8, and it is the only problem to have made the leap from that list to the Millennium Problem catalogue.

The hypothesis focuses on prime numbers. Prime numbers have always been elusive mathematical beasts. There is no formula that provides a list of all the prime numbers, and prime number tests, especially for large numbers, require calculations that demand exhaustive amounts of time. The mathematician Paul Erdös once said, in a paraphrase of a famous quotation by Albert Einstein, "God may not play dice with the Universe, but there's something strange going on with the prime numbers!"

The Riemann Hypothesis asserts that there is indeed something very strange going on with the primes—that there is a deep pattern hidden within the seemingly random distribution of prime numbers on the number line.

In the century and a half since it was first proposed, the Riemann Hypothesis has become a celebrity among unsolved problems. It made an appearance in the book (and movie) *A Beautiful Mind*, Sylvia Nasar's book about the life and work of Princeton mathematician John Nash. Apocalyptic news articles predict that a proof of the Riemann will bring Internet security, which depends on the apparently random distribution of prime numbers, crashing to its knees. Books have been written about its origins and the attempts to solve it.

Mathematicians have very good, experimental reasons to believe that the Riemann Hypothesis is true. Andrew Odlyzko, who worked on the first Cray computers in the 1970's, has carried out extensive calculations that deeply suggest the truth of the theorem. Such work provides verification, and verification reassures both mathematicians who work to prove its truth and mathematicians who act as if it is true.

Verification, however, is not proof. To *prove* a theorem means to provide a logical argument—within the structures assigned by mathematical rigor—that demonstrates, unequivocally, that the theorem is true for all cases. The statement "odd numbers are prime" is true for the numbers 3, 5 and 7, but in general, it is not true. The validity of the cases of 3, 5, and 7 do not prove the conjecture. (In fact, its falsity is obvious with consideration of the next odd number, 9.)

Carlson said he has no idea when a proof of the Riemann Hypothesis will emerge. He compares his predictive ability to that of a seismologist. "It's like an earthquake," he said of a solution to the problem. "Actually, I think we can predict earthquakes better than we can predict when it will be solved. You can say, this is a stress zones, and in 500 years something is going to happen, but you just don't know. Think of what's already happened, with Perelman. Nobody suspected what happened until it happened."

And yet least one mathematician, Dr. de Branges, believes that a proof of the Riemann Hypothesis already exists. He has posted it on his website, and it is accompanied by a long biographical statement, his "Apology," which details his personal, lifelong entanglement with the Riemann Hypothesis. De Branges is not a crank: he established himself in 1984 by presenting a proof of a significant unsolved problem known as the Bieberbach Conjecture.

There are parallels between Louis de Branges and Grigori Perelman, up to a point. De Branges, like Perelman, first submitted his work online. Also like Perelman, de

Branges bypassed the traditional channels of peer-review and community acceptance. But unlike Perelman, his work has failed to ignite any interest in the greater mathematics community. Perelman came from nowhere, dropped his proof of the Poincaré Conjecture into the middle of the world, and vanished. People believe it is true. De Branges, a well known member of the community, has insisted, time and time again, that he his proof of the Riemann Hypothesis is valid, but his claims fall on deaf ears. No one qualified to have an opinion seems to care.

One reason de Branges is being ignored is that the proof that he has posted online is actually the latest in a long series of proposed proofs. Each previous incarnation of his work fell apart under scrutiny, but he continues to insist that he's been unfairly ignored. He has organized numerous seminars at Purdue to look at his work, and nothing has come of it.

"If you claim to have proved a lot of proofs that have turned out to be wrong, and you haven't had the insight or intelligence to recognize your own errors, even when people point them out to you, your credibility goes down," said the Fields Institute's Keyfitz. "This is game theory, elementary game theory. There are only two moves here. You've already set up how the game's going to go."

Since only a small, international community is even competent enough to judge the validity of de Branges's proofs, it's not hard to imagine that they have grown tired of scrutinizing the work of their colleague. As his colleague's indifference has deepened, so has de Branges's sense of frustration, even outrage.

"Although I have been working towards a proof of the Riemann hypothesis for nearly fifty years, no colleague of mine has taken my work seriously," de Branges wrote in an email.

"I do not understand why mathematicians reject work on this problem. I can only testify that it exists," he wrote. Intellectually isolated, and increasingly upset about it, de Branges continues to work, mostly ignored.

The pursuit of mathematics can do funny things to people. Sophie Germain, an 18th century French mathematician, had to assume the identity of a man before her efforts were recognized. Germain proved a particular case of Fermat's last theorem, and the prime numbers that she identified have come to be called Germain primes. She, in turn, was inspired by Archimedes, whose death might be attributed to intense mathematical devotion.

During the second Punic War, as the story goes, Marcellus gave Roman soldiers free reign to pillage the city of Syracuse and slay any errant Greeks who had the misfortune to cross their paths. Any Greek, that is, except Archimedes, whose work Marcellus admired. One soldier, no doubt paying more attention to the first mandate than to the second, found himself in a home of an old man who crouched on the floor, drawing in the sand. Intensely absorbed in his figures, the man had barely noticed the war raging outside; nor had he noticed that his country had already lost. He did, however, become aware of the presence of the soldier – perhaps he saw a boot near his hand; perhaps the wind changed. He begged, 'Μὴ μου τοὺς κύκλους τάραττε.'—'Please sir, don't disturb my circles.'

The man in the sand was Archimedes. He died seconds later, and one can imagine that, in a strange sort of tragic resonance, it was appropriate that his blood should mingle with his work. You might argue, metaphorically, that in fact, his blood *was* his work. The very intensity that had brought him worldwide renown might arguably have contributed to his death. And who knows what the world lost when the flaws and misunderstandings of the so-called 'real world' bludgeoned his world of perfect abstractions?

Right now, groups of mathematicians are excavating the beauty of Perelman's proof, searching for errors, poking at holes, looking for spheres. Later this year, peer-reviewed books and articles should begin to appear, and once they do, the CMI can begin the countdown. In the meantime, perhaps Perelman will re-emerge. Perhaps he won't.

"He is an unusual fellow," Carlson said. "I don't really know exactly what motivates him. Creative people come in all stripes; some are exactly like Joe Smith on the street and others are quite unconventional. He's unconventional."

Why has Perelman dropped off the radar? Mathematicians offer various theories.

Perhaps Perelman believes that a million dollars is not enough. "You think about what society values. A basketball player dunks a basket and gets a 10 million dollar-a-year contract. You give this guy who solves this problem that's been around for 100 years a million dollars—that's cheap," said one mathematician who asked not to be named.

"The gossip that I heard," said another, is "that Perelman isn't going to publish his result because he thinks he'll get murdered by some Russians… If he got a million dollars, someone would kill him."

“I think that he feels that he has done his work, and maybe he has, maybe he’s moved on to other things,” said Carlson.

Even if his work turns out to have faults, Perelman has certainly drawn connections and introduced significant ideas that will advance the science. He probably has not done his work in pursuit of a million dollars—it’s a lousy way to get rich, after all—or even for the prestige of having published a solution.

Perhaps he did it for the sheer pleasure of solving a problem.

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